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Total correlations as multi-additive entanglement monotones

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Abstract

We propose general conditions for a measure of total correlations as an entanglement monotone using its pure and mixed convex-roof extension. In doing so, we derive crucial theorems and propose a concrete candidate for a total correlation measure which is a multi-additive multipartite entanglement monotone. We perform numerical simulations that show, comparatively, the dependence of the proposed coherent total correlation measure on the size of the register for several different quantum states and for global and pairwise correlation measures. We also show how to simplify some of the conditions required for monotonicity, in particular that a function must be non-increasing under local operations and classical communications.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum entanglement has long been recognized as the heart of the conceptual foundations of quantum theory [1–3]. More recently, it has been rediscovered as it provides a valuable tool such as a physical resource for applications in quantum computation [4, 5], information processing and communication [6, 7], phase transitions and many-body physics [8], and the very foundations of statistical mechanics and thermodynamics [9]. As a physical resource [10], the task of quantifying the amount of entanglement present in a given multipartite quantum state is a subject of the utmost importance [10]. This task is certainly not trivial, and even for the bipartite case there are issues which are not completely understood [11, 12]. The problem gets even more involved in the multi-qudit scenario, where many more challenges arise due to the structure of the product of Hilbert spaces [13].

In terms of the capability of certain states to perform a given computational task, the problem can be posed as to how to establish a hierarchy for the degree of entanglement of the given states. The concept of 'more entangled' in this particular sense is, however, quite relative, as it would depend on the task we want to perform. An alternative approach to following is that related to the development of axiomatic entanglement measures [14, 15], which tries to avoid the above feature. This approach establishes some basic properties that are to be satisfied and introduces some others that are desired, e.g., additivity or continuity. The 'additivity problem' is of relevance as it is related to various quantum information features such as channel capacity [12, 16, 17].

A measure of total correlations is, in general, different from a measure of total entanglement. For the case of pure states we can consider them equivalent; however, for mixed states they are two different concepts. In this work we ask the following: if we already have a total correlation measure $\mathcal{T}$, which for pure states is equivalent to an entanglement measure, can we find a general procedure to generate an entanglement measure out of it? What are the requirements to be fulfilled by $\mathcal{T}$? This is motivated by the bipartite case, where the entropy of one qubit is a measure of total correlations *and* entanglement for the pure state case, but requires a convex-roof to be able to quantify the entanglement. In the same spirit, here we consider a total correlation measure $\mathcal{T}$ and its convex roof $\mathcal{T}^*$ as a measure of entanglement and then find the conditions that have to be satisfied by them to become such an entanglement measure. We propose a general approach to multipartite entanglement measures starting from the concept of total correlations, seeking to generalize the entanglement of formation, which is in essence the quantum mutual information and its pure convex-roof construction [18]. We do this by establishing general conditions on total correlation functions and their pure and mixed convex-roof constructions so that they are multi-additive[4] entanglement measures. Surprisingly, this can be contextualized within just one main property which has only very recently been studied [19]. We then propose a particular candidate for a total correlation measure which captures all possible types of correlations and is consistent with additivity and strong super additivity.

The paper is organized as follows. We present the rationale of the work to be introduced by briefly stating some results derived in [20, 21]. Then, we generalize the procedure by putting it into a broader context—that of total correlation measures as entanglement measures. From this, we obtain general conditions and properties for total correlation measures that lead to an accurate quantification of entanglement, following the example of the entanglement of formation [11]. We give a specific candidate for the presented strategy and show that it satisfies the mentioned properties, thus obtaining a measure of entanglement that is fully multi-additive.

2. Background

We begin by summarizing some of the results previously introduced in [21] as a motivation for the construction of a general scheme. The calculations and proofs presented here are expected to contribute to a clearer understanding of the total correlation measure that we propose and test later in this work. In [21] we introduced a pairwise strategy to quantify quantum entanglement,

[4] This is to be differentiated from the usual bipartite additivity, for which $\rho^{ABCD} = \rho^{AB} \otimes \rho^{CD}$ implies $E(AC : BD) = E(A : B) + E(C : D)$. Here, we are concerned with the multipartite case, and as such, multi-additivity means that if $\rho^{ABCD} = \rho^{AB} \otimes \rho^{CD}$ then $E(A : B : C : D) = E(A : B) + E(C : D)$.

essentially seeking to generalize the entanglement of formation (EoF), which can be written as [11, 12]

$$E(\rho) = \min \sum p_i (1/2) I(A:B)(\rho_i), \tag{1}$$

where the minimization is over pure states. In the same spirit, we defined the measure

$$\mathcal{M}_{\mathcal{P}} = \sum_{(A,B)} \mathcal{P}(A,B), \tag{2}$$

where the sum is intended over all non-equivalent choices of the indices (A, B) and $\mathcal{P}$ is a probe quantity measuring the non-factorizability of a two-qudit density matrix ($\mathcal{P}(\rho_{AB}) = 0$ iff $\rho = \rho_A \otimes \rho_B$). The main candidate was then the mutual information [22]

$$\mathcal{P} = \tfrac{1}{2} I(A:B). \tag{3}$$

Note that for the case of two qudits, the measure is exactly the entanglement of formation. This was then extended to mixed states using the pure convex-roof construction [18]; thus

$$\mathcal{M}_{\mathcal{P}} = \min \sum p_i \sum_{(A,B)} \mathcal{P}(A,B)(\rho^i), \tag{4}$$

where the minimization is over every possible decomposition on pure states. We showed that this measure exhibits some relevant properties:

- For an N-qudit state, $\sum_{\{i,j\}} \frac{1}{2} I(i:j)$ is normalized to $(2-\delta_{2,N})^{-1} C_2^N \log_2 d$, where $I(i:j)$ denotes the von Neumann's quantum mutual information.
- The $|GHZ\rangle$ state is the state with the highest average of $\mathcal{P}$ over all possible pairs of qudits and *not* the state with the highest values for *all* possible pairs of qubits.

In addition, it was noted that there is a crucial property: if the measure reaches its minimum over pure state decompositions, i.e.

$$\sum_{A,B} \mathcal{P}(A,B)(\rho) \geqslant \min \sum p_i \sum_{A,B} \mathcal{P}(A,B)(\rho_i), \tag{5}$$

then this property simultaneously guarantees: (i) local operations and classical communication (LOCC) monotonicity, thus making the measure an entanglement monotone, (ii) *multi-additivity*: if $\rho^{1,\ldots,N} = \rho^{1,\ldots,m} \otimes \rho^{m+1,\ldots,N}$, then $\mathcal{M}_{\mathcal{P}}(1:2:\cdots:N-1:N) = \mathcal{M}_{\mathcal{P}}(\rho^{1,\ldots,m}) + \mathcal{M}_{\mathcal{P}}(\rho^{m+1,\ldots,N})$ and (iii) *multi-strong super additivity*: for arbitrary ρ, $\mathcal{M}_{\mathcal{P}}(1:\cdots:N) \geqslant \mathcal{M}_{\mathcal{P}}(\rho^{1,\ldots,m}) + \mathcal{M}_{\mathcal{P}}(\rho^{m+1,\ldots,N})$.

3. Generalizing the strategy

We now take this strategy a step further and emulate the entanglement of formation [11] in the two-qudit case, and apply the same argument to the multipartite case, i.e., we find a measure of total correlations $\mathcal{T}$ and use it to quantify the entanglement in pure states, and then, through its pure or mixed convex-roof [19] extension $\mathcal{T}^*$, extend it to mixed states.

We first establish the basic conditions that a measure of total correlations must fulfill. We stress that there are currently only some basic conditions [23–25] with no real consensus on more closed conditions having been achieved. The basic conditions that any total correlation function must satisfy are as follows:

TCF1. Positivity: $\mathcal{T}(\rho) \geqslant 0$.
TCF2. It vanishes on factorizable states only: $\mathcal{T}(\rho^{1,\ldots,N}) = 0$ iff $\rho^{1,\ldots,N} = \rho^1 \otimes \cdots \otimes \rho^N$.
TCF3. Invariance under ancillas: $\mathcal{T}(\rho \otimes (\bigotimes_i \sigma_i)) = \mathcal{T}(\rho)$.
TCF4. LU invariance.
TCF5. LO non-increasing.

We also require some properties which are natural to total correlation measures: $\mathcal{T}$ is multi-additive and multi-strongly sub-additive on mixed states, i.e., that the following properties hold:

(+) *Additivity* (ADD). Given two arbitrary states denoted by $\rho^{a_1,\ldots,a_r}$ and $\rho^{b_1,\ldots,b_s}$,

$$\mathcal{T}(\rho_A \otimes \rho_B) = \mathcal{T}(\rho_A) + \mathcal{T}(\rho_B). \tag{6}$$

(+) *Strong super additivity* (SSA). Given a generic N-partite state $\rho^{1,\ldots,N}$,

$$\mathcal{T}(\rho^{1,\ldots,N}) \geqslant \mathcal{T}(\rho^{1,\ldots,m}) + \mathcal{T}(\rho^{m+1,\ldots,N}). \tag{7}$$

This is a natural condition to ask for, for when we make a partition on the state we immediately destroy correlations and thus justify the inequality.

Note that the above properties involve no convex roofs and are demanded over the quantity under consideration, which implies that the measure must be applicable to density matrices and not only to pure states. We want, now, to generalize equation (5) to this scenario, and we also want to take into account the possibility of mixed state convex-roof extensions.

Pure (mixed) convex-roof consistent (PCRC (MCRC)). This is the generalization of equation (5) and is the requirement that the convex-roof minimization is attained on decompositions over pure (general) states, which is equivalent to

$$\mathcal{T}(\rho) \geqslant \mathcal{T}^*(\rho) = \min \sum p_a \mathcal{T}(\rho_a), \tag{8}$$

where the minimization is intended over pure (general) state decompositions of the state ρ. It is interesting to observe that a mixed state convex-roof immediately guarantees the MCRC condition. We note that recently Synak–Radke and Horodecki [19] have described a process termed *arrowing* which generalizes the convex-roof construction and showed that the convex roof of a function inherits the properties of the function, such as the continuity and the asymptotic continuity. Also note that the pure-convex roof is a particular case of the mixed convex-roof.

A total correlation measure that satisfies the above conditions shall be referred to as a *complete total correlation measure*. We show next that a measure of total correlations satisfying the above-mentioned conditions leads to a multi-additive measure of entanglement, provided that it also satisfies the monotonicity conditions. For notation purposes and for the sake of clarity, we emphasize that whenever we write $F(\rho^X)$ we consider each qudit in the register X to be independent from the others, i.e., $F(\rho^X) = F(x_1 : \cdots : x_N)$. By emulating the two-qudit case, we introduce

Definition 3.1. *A total correlation measure $\mathcal{T}$ is a coherent total correlation measure if the maximally quantum correlated state has a higher value than the maximally classically correlated state.*

4. Total correlations as multi-additive entanglement monotones

We next show the benefits of the strategy, proving two basic theorems which are strongly supported on the PCRC (MCRC) condition.

Theorem 4.1. *Let $\mathcal{T}$ be a measure of total correlations on pure states and let $\mathcal{T}^*$ be its pure (mixed) convex-roof extension. If $\mathcal{T}$ satisfies ADD, SSA and PCRC (MCRC) then it is a multi-additive and multi-strongly super additive quantity. We say it is also a multi-additive entanglement measure if it is also an entanglement monotone.*

Proof. We consider two qudit registers, namely $\{u_s\}$ and $\{v_r\}$. Let ρ^U and ρ^V be the corresponding qudit density matrices with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)}\sigma_a^{(i)}$, such that $\rho = \rho^U \otimes \rho^V$. We first consider an arbitrary non-bifactorizable decomposition assuming that it is the minimum of the convex roof, and then show that a bifactorizable decomposition is a lower bound for any possible non-bifactorizable decomposition:

$$\begin{aligned}\mathcal{T}^*(u_1:\cdots:u_s:v_1:\cdots:v_r)(\rho^{UV}) &= \sum q_a \mathcal{T}\left(\rho_a^{UV}\right)\\ &\geqslant \sum q_a\left(\mathcal{T}\left(\rho_a^U\right)+\mathcal{T}\left(\rho_a^V\right)\right)\\ &= \sum q_a\left(\mathcal{T}\left(\rho_a^U\right)\right)+\sum q_a\left(\mathcal{T}\left(\rho_a^V\right)\right)\\ &\geqslant \sum q_a \mathcal{T}^*\left(\rho_a^U\right)+\sum q_a\mathcal{T}^*\left(\rho_a^V\right)\\ &\geqslant \mathcal{T}^*(\rho^U)+\mathcal{T}^*(\rho^V).\end{aligned} \tag{9}$$

We emphasize that $\mathcal{T}$ corresponds to a total correlation measure which has to be defined for any density operator and, not being an entanglement measure for mixed states, needs no convex-roof techniques. Thus, the conditions ADD (equation (6)) and SSA (equation (7)) for total correlation measures are valid for $\mathcal{T}$. So, in equation (9), from lines 1 to 2 we have used equation (7); from lines 2 to 3 we just rewrite; from lines 3 to 4 we use equation (8) and, finally, from lines 4 to 5 we have used the convexity of $\mathcal{T}^*$. It is important to stress here the relevance of equation (8), not only in the procedure that leads to equation (9) but also for most of the deductions along this paper (e.g., for equation (13)).

Multi-strong super additivity, i.e. $\mathcal{T}^*(\rho^{1,\ldots,N}) \geqslant \mathcal{T}^*(\rho^{1,\ldots,m}) + \mathcal{T}^*(\rho^{m+1,\ldots,N})$, is demonstrated using the same reasoning as above, but with identifications $\rho = \rho^{1,\ldots,N}$, $\rho_1 = \rho^{1,\ldots,m}$ and $\rho_2 = \rho^{m+1,\ldots,N}$. The last set of equations would then read

$$\mathcal{T}^*(\rho^{1,\ldots,N}) \geqslant \mathcal{T}^*(\rho^{1,\ldots,m}) + \mathcal{T}^*(\rho^{m+1,\ldots,N}). \tag{10}$$

□

Theorem 4.2. *Any complete total correlation measure $\mathcal{T}$ extended to mixed states through the pure (mixed) convex-roof construction $\mathcal{T}^*$ is an entanglement monotone.*

Proof. We only need to prove that a measure defined in this way is an LOCC non-increasing function, as the other properties are provided by the hypothesis. In doing so, we will make use of the FLAGS conditions introduced in [27]: an entanglement measure E is a monotone if and only if it is a local unitary invariant and satisfies

$$E\left(\sum p_i\rho_i\otimes|i\rangle\langle i|\right) = \sum p_i E(\rho_i). \tag{11}$$

To this end, we proceed in the following way. First, by convexity and multi-additivity, we have

$$\mathcal{T}^*\left(\sum p_i\rho_i\otimes|i\rangle\langle i|\right) \leqslant \sum p_i\mathcal{T}^*(\rho_i\otimes|i\rangle\langle i|) = \sum p_i\mathcal{T}^*(\rho_i), \tag{12}$$

as one qudit has no entanglement. Now we must show that $\mathcal{T}^*(\tilde{\rho} = \sum p_i\rho_i\otimes|i\rangle\langle i|) \geqslant \sum p_i\mathcal{T}^*(\rho_i)$ to get a full equality. To do this, we must show that the optimal decomposition of $\tilde{\rho}$ is bounded by $\sum p_i\mathcal{T}^*(\rho_i)$. As in previous cases, let us assume that the minimal decomposition is given by $\rho = t_a\eta_a^{SR}$, where S may contain any number of qudits and R contains a single qudit. Then

$$\begin{aligned}\mathcal{T}^*(\rho) &= \sum t_a\mathcal{T}\left(\eta_a^{SR}\right)\\ &\geqslant \sum t_a\left(\mathcal{T}\left(\eta_a^S\right)+\mathcal{T}\left(\eta_a^R\right)\right) \qquad \text{(by SSA)}\end{aligned}$$

$$\geqslant \sum t_a \left(\mathcal{T}^*\left(\eta_a^S\right) + \mathcal{T}^*\left(\eta_a^R\right)\right) \quad \text{(by PCRC)}$$
$$= \sum t_a \left(\mathcal{T}^*\left(\eta_a^S \otimes \eta_a^R\right)\right) \quad \text{(by full multi-ADD)}$$
$$= \sum t_a \mathcal{T}^*\left(\eta_a^S\right), \tag{13}$$

where the last line follows as one qudit has no entanglement, i.e., $\mathcal{T}^*(\rho^{\dim(R)=1}) = 0$. This shows that for an arbitrary decomposition $\rho^{SR} = p_i \rho_i^{SR}$, assumed to minimize $\mathcal{T}^*, \mathcal{T}^*(\rho^{SR}) \geqslant \sum t_a \mathcal{T}^*\left(\eta_a^S \otimes \eta_a^R\right) = \sum t_a \mathcal{T}^*\left(\eta_a^S\right)$, in particular a decomposition of the form $\tilde{\rho} = \sum p_i \rho_i \otimes |i\rangle\langle i|$ (with $\eta_a^R = |a\rangle\langle a|$), which exists by the hypothesis, is a lower bound for it. However, we also showed in the previous step that for such a decomposition $\mathcal{T}^*(\rho^{SR}) \leqslant \sum t_a \mathcal{T}^*\left(\eta_a^S \otimes |a\rangle\langle a|^R\right) = \sum t_a \mathcal{T}^*\left(\eta_a^S\right)$, thus

$$\mathcal{T}^*\left(\rho^{SR} = \sum t_a \eta_a^S \otimes |a\rangle\langle a|^R\right) = \sum t_a \mathcal{T}^*\left(\eta_a^S\right), \tag{14}$$

as claimed. □

5. Building multi-additive entanglement measures: a complete and coherent total correlation measure

We next analyze some issues related to the monotonicity of the proposed measure $\mathcal{M}$. For this to be a monotone, equation (5), which is the analog of PCRC, must be satisfied. In this section, we shall consider the mixed convex-roof extension of such a function, which by the theorems above is a multi-additive entanglement monotone. Note that although $\mathcal{M}$ is not coherent, this does not pose a setback, as this simply means that it does not fully account for the quantification of certain types of correlations which are characteristics of a total correlation measure.

We now build a *coherent* and *complete* total correlation measure which considers all possible correlations while maintaining additivity and strong super additivity. There are several possibilities of correlations in a state.

(i) *Pairwise total correlations.* As shown in section 2, they are additive and strongly super additive. (ii) *Bipartite correlations*. Consider an N-qubit state and an arbitrary bipartition $\mathcal{B} = S(\rho_R) + S(\rho_{N-R}) - S(\rho_N)$. This quantity is strongly super additive but not additive in general. (iii) *Correlations among subsets.* This can be considered as a general case containing the bipartite correlations, and the pairwise correlations, however their sum $\mathcal{L}$, although being strongly super additive, is not additive in general. (iv) *Single-qubit correlations or global correlations*. We consider single-qubit entropies as global correlations. The correlations for this case are given by

$$\begin{aligned}\mathcal{O}(\rho_{1,\ldots,N}) &= \frac{1}{2}\left(\sum S(\rho_i) - S(\rho_{1,\ldots,N})\right)\\ &\geqslant \frac{1}{2}\left(\sum S(\rho_i) - S(\rho_{1,\ldots,m}) - S(\rho_{m+1,\ldots,N})\right)\\ &= \mathcal{O}(\rho_{1,\ldots,m}) + \mathcal{O}(\rho_{m+1,\ldots,N}),\end{aligned} \tag{15}$$

thus strong super additivity is guaranteed. Additivity on pure states follows analogously. This measure has been studied previously [23–25] and it has been proven to measure the basic properties and to be coherent. It fails, however, in discriminating between $|EPR\rangle \otimes |EPR\rangle$ and $|GHZ\rangle_4$ states, so we can say again that it does not succeed in quantifying certain total correlations.

Note that (i) and (ii) are particular cases of the subset case, $\mathcal{L}$, and when we sum over all possible choices of subsets we get a strongly super additive quantity as a total correlation measure (SSA); this is, however, not additive. To see this, we note that when any artificial bipartition is performed on an arbitrary density matrix $\rho \rightarrow \rho_P, \rho_{\bar{P}}$, $\mathcal{L}(\rho)$ contains all positive terms in $\mathcal{L}(\rho_P) + \mathcal{L}(\rho_{\bar{P}})$, thus evidencing strong super additivity. However, when there is a natural bipartition, namely $\rho = \rho_P \otimes \rho_{\bar{P}}$, non-trivial multiplicities of the elements of $\mathcal{L}(\rho_P)$ and $\mathcal{L}(\rho_{\bar{P}})$ appear, and so there is no additivity.

With this in mind, we now build a coherent and complete total correlation function $\mathcal{S}$ and its mixed convex-roof extension $\mathcal{S}^*$, which by the above theorems is an entanglement measure, as

$$\mathcal{S} = \frac{\mathcal{O} + \mathcal{M}}{2}, \tag{16}$$

which is just the average of the two types of correlations which maintain additivity and strong super additivity. Their sum helps us to overcome the issues they posed separately, at the cost of making the PCRC conjectured condition weaker as $\mathcal{S} - \mathcal{M}|_{\text{on pure states}} \geqslant \mathcal{S} - \mathcal{M}|_{\text{on mixed states}}$.

Given the bounds for each measure, it is easy to see that

$$\mathcal{S} \leqslant \frac{\left(C_2^N(2 - \delta_{N,2})\right)^{-1} + N/2}{2} \log_2 d. \tag{17}$$

We can alternatively write our measure in terms of relative entropies as

$$\mathcal{S} = \frac{\sum_{i \leqslant j} S(\rho_{ij} \| \rho_i \otimes \rho_j) + S(\rho_{1,\ldots,N} \| \rho_1 \otimes \cdots \otimes \rho_N)}{4}, \tag{18}$$

which immediately suggests the continuity of our measure. This is, however, not surprising as this measure is written in terms of the quantum mutual information and relative entropies, which are asymptotically continuous. Also note that if two N-partite density matrices are close, then their reduced density matrices are also close. The maximum of the measure is again attained by the $|GHZ\rangle$ state.

The benefits of using $\mathcal{S}$ instead of resorting to $\mathcal{O}$ or $\mathcal{M}$ separately are as follows. We capture more types of correlations than by means of $\mathcal{O}$ alone, which has generally been used as a measure of total correlations [24, 25]. Consider, for example, the case of the $|GHZ\rangle$ state, the cluster state [28] and the $|EPR\rangle \otimes |EPR\rangle$ state, which exhibit different types of correlations: our measure can effectively distinguish all of them. Perhaps the most notable comparison is that between the cluster state and the $|EPR\rangle \otimes |EPR\rangle$ state: $\mathcal{O}$ alone fails to distinguish among them, while $\mathcal{S}$ makes the differentiation due to the inclusion of the pairwise total correlations. It is not difficult to see that $\mathcal{S}$, through its construction, captures more types of correlations, and thus is a more complete measure of total correlations. Also note that $\mathcal{S}$ considers all types of correlations consistent with additivity and strong super additivity.

5.1. Comparing $\mathcal{O}$ and $\mathcal{S}$

We now quantify the above established comparison between the measures $\mathcal{O}$ and $\mathcal{S}$, and analyze the behavior of specific cases in terms of the number of particles. Before we proceed to consider the concrete cases, we note that the measure $\mathcal{O}$ is the von Neumann analog of the Meyer–Wallach–Brennen measure $\mathcal{MW}$ [29, 30], namely the structure is the same but with von Neumann's entropy instead of the linear entropy. Furthermore, in our multipartite entanglement measure scenario $\mathcal{O}^* = \mathcal{MW}^*$.

We next define some of the states we investigate below:

(i) $|GHZ\rangle_N = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$,

(ii) $|\text{Cluster}\rangle_N = \frac{1}{\sqrt{N}}(|0\rangle^{\otimes N} + |0\rangle^{\otimes N/2}\,|1\rangle^{\otimes N/2} + |1\rangle^{\otimes N/2}\,|0\rangle^{\otimes N/2} - |1\rangle^{\otimes N})$,

(iii) $|W\rangle_N = \frac{1}{\sqrt{N}}\,(|10\ldots0\rangle + |010\ldots0\rangle + \cdots + |0\ldots01\rangle)$,

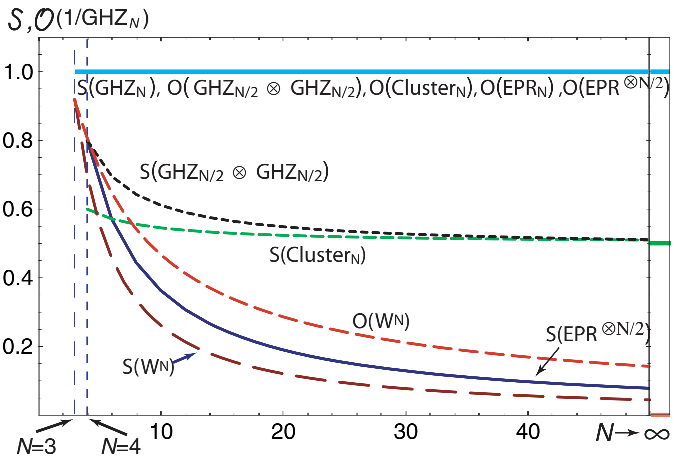

Figure 1. *N-dependence relative entanglement comparative graph.* The entanglement of all states is compared to the entanglement of the *GHZ* state, thus we plot the relative entanglement of each state. Note how $\mathcal{O}$ fails to discriminate among several kinds of states, while $\mathcal{S}$ does indeed establish a hierarchy in the degree of entanglement of the different states.

(iv) $|\bar{W}\rangle_N = \frac{1}{\sqrt{N}}\,(|01\ldots1\rangle + |101\ldots1\rangle + \cdots + |1\ldots10\rangle)$,

(v) $|EPR\rangle = |GHZ\rangle_2$.

First, we compare the results of the application of both measures to known states and then characterize their dependence on the size of the quantum register. This is first plotted in figure 1, where $\mathcal{S}$ and $\mathcal{O}$ appear evaluated for several different states as a function of the number of particles N. We note that as $\mathcal{O}$ relies on single-qubit von Neumann's entropies, it fails to distinguish among several states, as shown by the horizontal, solid line (in blue) of the figure. It is interesting to note that in the infinite qudit *thermodynamic* limit, the cluster and the $|GHZ\rangle_{N/2}^{\otimes 2}$ have the same value of $\mathcal{O}$. This is so because their pairwise correlation structure is the same, namely the same pairwise total correlations vanish (permutations are of course analog) with different values, and the global correlations compensate for this difference in such a way that they yield the same limit. The graph also shows the distinguishability advantages of our measure $\mathcal{S}$ when applied to all of the above-introduced quantum states.

In the same comparative spirit, we now proceed to analyze two cases of particular interest. In the first case, plotted in figure 2 for the state $|\psi\rangle_x^{(1)} = \sqrt{x}\,|GHZ\rangle_N + \sqrt{1-x}\,|W\rangle_N$ as a function of the parameters x and N, there is no major difference between the global ($\mathcal{O}$), pairwise ($\mathcal{M}$) and the total measure ($\mathcal{S}$), only a small quantitative discrepancy in their values as a function of x and N. Thus, we note that for some states, the behavior of the different measures is quite similar, i.e., for some states the contribution due to the pairwise correlations is not very significant.

The next case, however, evidences the existence of states for which the pairwise contributions become of particular relevance. This case poses two main features: (i) pairwise correlations become important and (ii) entanglement or total correlations rise with the number of qubits for a range of x values, as can be seen in figure 3. In the thermodynamic limit, the state $|\psi\rangle_{x=1/2}^{(2)} = \frac{1}{\sqrt{2}}(|W\rangle_N + |\bar{W}\rangle_N)$ has the same entanglement as the *GHZ* state. This is very interesting as, in principle, only in the thermodynamic limit would one have enough

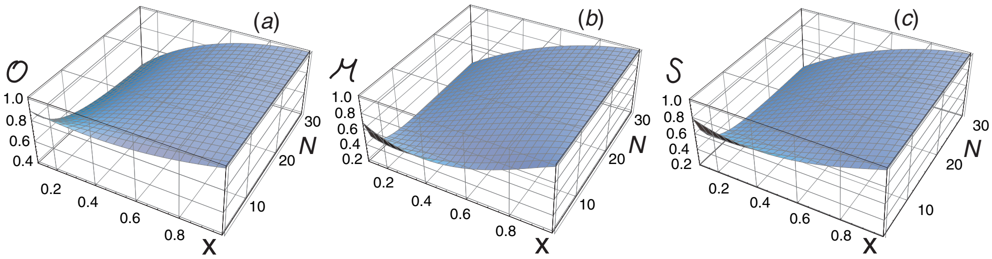

Figure 2. (a) Global, (b) pairwise and (c) total correlation measures for the family of states $|\psi\rangle_x^{(1)} = \sqrt{x}\,|GHZ\rangle_N + \sqrt{1-x}\,|W\rangle_N$, as a function of the parameters x and N. As in figure 1, the graph is normalized by the value for the GHZ state.

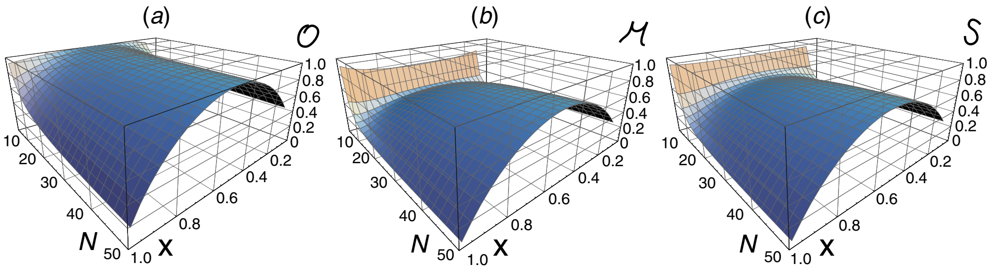

Figure 3. (a) Global, (b) pairwise and (c) total correlation measures for the family of states $|\psi\rangle_x^{(2)} = \sqrt{x}\,|W\rangle_N + \sqrt{1-x}\,|\bar{W}\rangle_N$, as a function of the parameters x and N. As in figure 1, the graph is normalized by the value for the GHZ state.

degrees of freedom to perform local unitary operations to transform one state into the other thus justifying the equality.

Through this, we have shown how total correlation measures can generate multi-additive entanglement monotones using their pure (mixed) convex-roof extension to mixed states. In doing so, we have also found the relevant conditions for the case of pure or mixed convex-roof extensions.

As a main result, we would like to stress that, using the mixed convex-roof extension, *the proposed total correlation measure $\mathcal{S}$ is a complete coherent total correlation measure as well as a multi-additive entanglement monotone*.

6. Simplifying axiomatic entanglement measures

The above theorems can be specialized to the case where we restrict ourselves to pure convex-roof extensions in order to get simpler and more flexible conditions for building fully multi-additive multipartite entanglement monotones. The following theorem states such a result.

Theorem 6.1. *Let $\mathcal{E}$ be a quantity defined from density matrices to real numbers, and let $\mathcal{E}^*$ be its pure convex-roof extension. $\mathcal{E}^*$ is a fully multi-additive and multi-strongly super additive entanglement monotone if it satisfies the following properties (α-conditions):*

FAEM1. *Vanishing on separable pure states:*
$\mathcal{E}(|\Psi\rangle_1 \otimes \cdots \otimes |\Psi\rangle_N) = 0.$

FAEM2. *The existence of maximally entangled states: There exist pure states* $|\phi\rangle$ *such that* $\mathcal{E}(|\phi\rangle) \geqslant \mathcal{E}(|\Psi\rangle)$ *for all* $|\Psi\rangle$.

FAEM3. $\mathcal{E}$ *is invariant under local unitary (LU) operations.*

FAEM4. *Strongly super additive on pure states:* $\mathcal{E}(|\Psi\rangle\langle\Psi|) \geqslant \mathcal{E}(\mathrm{Tr}_S[|\Psi\rangle\langle\Psi|]) + \mathcal{E}(\mathrm{Tr}_{\bar{S}}[|\Psi\rangle\langle\Psi|])$.

FAEM5. *Additive on pure states: If* $|\Psi\rangle = |\psi\rangle_{\bar{S}} \otimes |\psi\rangle_S$, *then* $\mathcal{E}(|\Psi\rangle) = \mathcal{E}(|\psi\rangle_{\bar{S}}) + \mathcal{E}(|\psi\rangle_S)$.

FAEM6. *Pure convex-roof consistent:*

$$\mathcal{E}(\rho) \geqslant \mathcal{E}^*(\rho) = \min \sum_i p_i \mathcal{E}(|\Psi^i\rangle\langle\Psi^i|). \tag{19}$$

Note that if $\mathcal{E}$ is concave then equation (19) is automatically satisfied.

Proof.

(i) *Full multi-additivity and multi-strong super additivity.* We initially consider the four-partite case, and then extend the argument to the N-partite system. Let us consider the two two-qudit density matrices $\rho^{(1)}$ and $\rho^{(2)}$ with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)} \sigma_a^{(i)}$, such that $\rho = \rho^{(1)} \otimes \rho^{(2)}$. We first consider an arbitrary non-bifactorizable decomposition and then show that this must have higher values for the convex-roof extension than those for the bifactorizable decomposition, thus showing that the latter decomposition is indeed the real minimum for the pure convex-roof construction:

$$\begin{aligned}
\mathcal{E}^*(\rho) &= \sum q_a \mathcal{E}\left(\rho_a^{1234}\right) \\
&\geqslant \sum q_a \left(\mathcal{E}\left(\rho_a^{12}\right) + \mathcal{E}\left(\rho_a^{34}\right)\right) \qquad \text{(by FAEM4)} \\
&= \sum q_a \left(\mathcal{E}\left(\rho_a^{12}\right)\right) + \sum q_a \left(\mathcal{E}\left(\rho_a^{34}\right)\right) \\
&\geqslant \sum q_a \left(\min \sum_s u_s^{(a)} \mathcal{E}\big(\rho_s^{(a)12}\big)\right) + \text{i.d. over } \{34\} \\
&\geqslant \mathcal{T}^*\big(\rho^{(1)}\big) + \mathcal{T}^*\big(\rho^{(2)}\big),
\end{aligned} \tag{20}$$

where the last inequality follows as the decomposition resulting from minimizing every mixed density matrix in the expansion may not be the actual minimal decomposition of the complete matrix, i.e.,

$$\begin{aligned}
r_1 \mathcal{E}(\eta^{(1)}) + r_2 \mathcal{E}(\eta^{(2)}) &\geqslant \sum r_1 \left(\min \sum_s u_s^{(1)} \mathcal{E}\big(\eta_s^{(1)}\big)\right) + \text{i.d. over } \{2\} \\
&\geqslant \mathcal{E}^* \left(\sum r_c \eta_c\right).
\end{aligned} \tag{21}$$

For the N-partite case, we follow a similar line of reasoning as above. Here we consider two qudit registers, namely $\{U\}$ and $\{V\}$. Let u and v be the corresponding qudit density matrices $\rho^{(1)}$ and $\rho^{(2)}$ with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)} \sigma_a^{(i)}$, such that $\rho = \rho^U \otimes \rho^V$. Using the same reasoning as previously, we first consider an arbitrary non-bifactorizable decomposition and then show that a bifactorizable decomposition is a lower bound for any possible non-bifactorizable decomposition:

$$\begin{aligned}
\mathcal{E}^*(\rho^{UV}) &= \sum q_a \mathcal{E}\left(\rho_a^{UV}\right) \\
&\geqslant \sum q_a \left(\mathcal{E}\left(\rho_a^{U}\right) + \mathcal{E}\left(\rho_a^{V}\right)\right) \qquad \text{(by FAEM4)} \\
&= \sum q_a \left(\mathcal{E}\left(\rho_a^{U}\right)\right) + \sum q_a \left(\mathcal{E}\left(\rho_a^{V}\right)\right)
\end{aligned}$$

$$\geqslant \sum q_a \mathcal{E}^*\left(\rho_a^U\right) + \sum q_a \mathcal{E}^*\left(\rho_a^V\right)$$
$$\geqslant \mathcal{E}^*(\rho^U) + \mathcal{E}^*(\rho^V), \tag{22}$$

where the last inequality follows from the same argument that led to equations (21).

SSA, i.e. $\mathcal{E}^*(\rho^{1,\ldots,N}) \geqslant \mathcal{E}^*(\rho^{1,\ldots,m}) + \mathcal{E}^*(\rho^{m+1,\ldots,N})$, is demonstrated using the same procedure as above, but with identifications $\rho = \rho^{1,\ldots,N}$, $\rho_1 = \rho^{1,\ldots,m}$ and $\rho_2 = \rho^{m+1,\ldots,N}$. The last set of equations would then read

$$\mathcal{E}^*(\rho^{1,\ldots,N}) \geqslant \mathcal{E}^*(\rho^{1,\ldots,m}) + \mathcal{E}^*(\rho^{m+1,\ldots,N}). \tag{23}$$

(ii) *Monotonicity.* We only need to prove that a measure defined in this way is an LOCC non-increasing function, as the other properties are provided by the hypothesis. In doing so, we will make use of the FLAGS conditions [27]: first, by convexity and FAEM5, we have

$$\mathcal{E}^*\left(\sum p_i\rho_i \otimes |i\rangle\langle i|\right) \leqslant \sum p_i \mathcal{E}^*(\rho_i \otimes |i\rangle\langle i|) = \sum p_i \mathcal{E}^*(\rho_i). \tag{24}$$

Now we must show that $\mathcal{E}^*\big(\sum p_i\rho_i \otimes |i\rangle\langle i|\big) \geqslant \sum p_i \mathcal{E}^*(\rho_i)$ to get a full equality. To do this, we must show that the optimal decomposition of $\tilde{\rho} = \sum p_i\rho_i \otimes |i\rangle\langle i|$ is bounded by $\sum p_i\mathcal{E}^*(\rho_i)$. Note that the above decomposition of $\tilde{\rho}$ implies that there exists a decomposition in pure states of the form

$$\rho = \sum_s \sum_i q_s p_i^{(s)} \big|\Psi_i^{(s)}\big\rangle\big\langle\Psi_i^{(s)}\big| \otimes |s\rangle\langle s|, \tag{25}$$

which is valid as $\sum_{s,i} q_s p_i^{(s)} = 1$. We now show that if such a decomposition exists, then it minimizes $\mathcal{E}^*(\rho)$. As in previous cases, let us assume that the minimal decomposition is given by $\rho = t_a\,|\psi\rangle_a^{SR}\,\langle\psi|_a^{SR} = t_a\eta_a^{SR}$, where S may contain any number of qudits and R contains a single qudit. Then

$$\begin{aligned}
\mathcal{E}^*(\rho) &= \sum t_a \mathcal{E}\left(\eta_a^{SR}\right)\\
&\geqslant \sum t_a\left(\mathcal{E}\left(\eta_a^S\right) + \mathcal{E}\left(\eta_a^R\right)\right) && \text{(by SSA)}\\
&\geqslant \sum t_a\left(\mathcal{E}^*\left(\eta_a^S\right) + \mathcal{E}^*\left(\eta_a^R\right)\right) && \text{(by PCRC)}\\
&= \sum t_a\left(\mathcal{E}^*\left(\eta_a^S \otimes \eta_a^R\right)\right) && \text{(by full ADD)}\\
&= \sum t_a \mathcal{E}^*\left(\eta_a^S\right),
\end{aligned} \tag{26}$$

where the last line follows as one qudit has no entanglement, i.e., $\mathcal{E}^*(\rho^{\dim(R)=1}) = 0$. This shows that for an arbitrary decomposition $\left(\rho^{SR} = p_i\rho_i^{SR}\right)$ assumed to minimize $\mathcal{E}^*$, $\mathcal{E}^*\left(\rho_i^{SR}\right) \geqslant \sum t_a\mathcal{E}^*\left(\eta_a^S\otimes\eta_i^R\right) = \sum t_a\mathcal{E}^*\left(\eta_a^S\right)$ that is a decomposition of the form of equation (25), which exists by the hypothesis, is a lower bound for it. However, we also showed in the previous step that for such a decomposition, $\mathcal{E}^*(\rho^{SR}) \leqslant \sum t_a\mathcal{E}^*\left(\eta_a^S\otimes\eta_a^R\right) = \sum t_a\mathcal{E}^*\left(\eta_a^S\right)$, thus

$$\mathcal{E}^*\left(\rho^{SR} = \sum t_a\eta_a^S \otimes \eta_a^R\right) = \sum t_a\mathcal{E}^*\left(\eta_a^S\right), \tag{27}$$

as claimed. Note that this demonstration is even more general, as it does not require that $\eta_a^R = |a\rangle\langle a|$, only that it be a single-qudit density matrix (FLAG). □

With this result, we simplify the monotonicity conditions for axiomatic entanglement measures as in general the LOCC non-increasing character of a function is not trivial to prove. As an example, it easily follows from the theorem that the Meyer–Wallach–Brennen measure using $\mathcal{E} = \sum S(\rho_i) \equiv \mathcal{MW}$ [29, 30] and its pure convex-roof extension is a multi-additive entanglement monotone.

7. Conclusions

We have proposed a strategy for quantifying entanglement in the multipartite case, based on measures of total correlations and its pure convex-roof extension. Within a natural scenario, we have demonstrated that these total correlation measures are entanglement monotones. Furthermore, we have proposed a specific quantity to simultaneously fulfill the role of a total correlation measure and a multi-additive entanglement monotone.

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